Consider the curve $\beta(t) = (t+3, t^2-1, 2)$. (2.9)

(a) Describe the shape of this curve.

(b) Find the unit tangent vector of this curve.

(c) Find the acceleration vector.

(d) Find the tangential component of the acceleration vector.

(e) For what intervals in t is the speed of β increasing?

(f) Find the arclength of β between $t=-1$ and $t=2$.

(2.9) (a) parabola within plane $z=2$.

(b) $T(t) = \frac{1}{\|\beta'\|}\beta'(t)$

$$\beta'(t) = (1, 2t, 0)$$

$$\text{speed} = \|\beta'(t)\| = \sqrt{1+4t^2+0}$$

unit tangent

$$= \sqrt{1+4t^2}.$$

$$T(t) = \frac{1}{\sqrt{1+4t^2}}(1, 2t, 0)$$

$$= \boxed{\left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}, 0\right)}.$$

$$= \frac{1}{\sqrt{1+4t^2}}\hat{\imath} + \frac{2t}{\sqrt{1+4t^2}}\hat{\jmath}$$

(c) $\beta''(t) = (0, 2, 0)$.

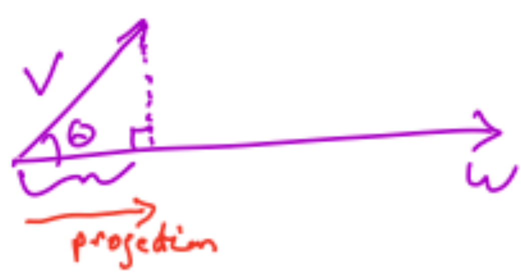
(d) What's $a_T(t)$ = tangential component of acceleration = projection of the acceleration vector onto the velocity vector.

$$\left(\begin{array}{c}\text{Projection of} \\ \text{a vector } v \\ \text{onto a vector} \\ w\end{array}\right) = \frac{\langle v, w\rangle}{\langle w, w\rangle} w$$

Why? $\langle v, w\rangle = v \cdot w = \|v\| \, \|w\| \cos\theta$

$\langle w, w\rangle = w \cdot w = \|w\|^2$

$$\frac{\langle v, w\rangle}{\langle w, w\rangle} w = \frac{\|v\| \, \cancel{\|w\|} \cos\theta}{\cancel{\|w\|} \, \|w\|} w = \left(\|v\| \cos\theta\right) \underbrace{\frac{1}{\|w\|} w}_{\text{unit vector in direction of } w}$$



$\|v\| \cos\theta$

$\frac{\text{adj}}{\text{hyp}} = \cos\theta$

$\text{adj} = (\cos\theta)\,\text{hyp} = \|v\| \cos\theta$

We want $a_T(t)$ = proj. of $\beta''(t)$ onto $\beta'(t)$

$$a_T(t) = \frac{\langle \beta'' \cdot \beta' \rangle}{\langle \beta' \cdot \beta' \rangle} \beta'$$

$$= \frac{(0,2,0)\cdot(1,2t,0)}{(1,2t,0)\cdot(1,2t,0)} (1,2t,0)$$

$$= \frac{4t}{1+4t^2}(1,2t,0)$$

$$= \boxed{\left(\frac{4t}{1+4t^2}, \frac{8t^2}{1+4t^2}, 0\right)}$$ tangential component of acceleration

Note: 0 at $t=0$, to the right if $t>0$
to the left if $t<0$

(e) When is the speed increasing?

When $a_T(t)$ is in the same direction as the velocity $\beta'(t)$

$\iff$ when $\beta'' \cdot \beta' > 0$

ie $\cos\theta > 0$ ie θ = angle between β'' & β' is $< \pi/2$.

$$\beta''\cdot\beta' = (0,2,0)\cdot(1,2t,0) = 4t > 0$$

$$\iff \boxed{t>0}.$$

$$\iff \boxed{t \in (0,\infty)}$$

(f) Length of β between $t=-1$ and $t=2$

$$\text{arclength} = \int_{-1}^{2} \underbrace{\|\beta'(t)\|}_{\text{speed}}\, \underbrace{dt}_{\substack{\Delta t \\ \text{time}}}$$

$$= \int_{-1}^{2} \sqrt{1+4t^2}\, dt \approx \boxed{6.126}$$

The gradient & contour diagrams

$F(x,y)$ function

$$\nabla F = (F_x, F_y) \leftarrow \text{example of a vector field}$$

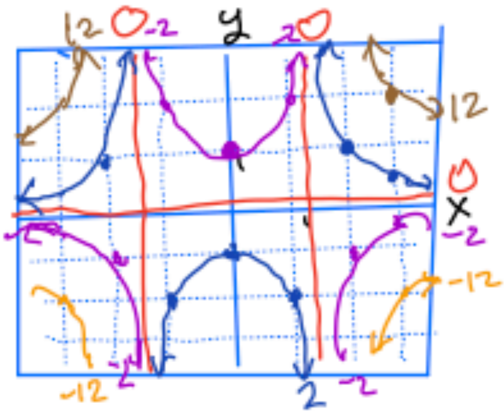
(in higher dimensions — there would just be more variables

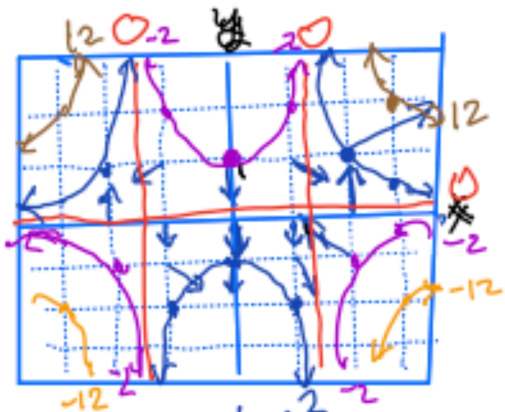
$$g(x_1,x_2,x_3,x_4) \Rightarrow \nabla g = (g_{x_1}, g_{x_2}, g_{x_3}, g_{x_4})$$

vector field: at each (x,y), we get a vector.

Example: Let $f(x,y) = x^2y - 2y$

$$\nabla f = (2xy,\ x^2-2)$$





Contours:

$$x^2y - 2y = 0$$

$$(x^2-2)y = 0$$

$$(x+\sqrt{2})(x-\sqrt{2})y = 0$$

$$x^2y - 2y = 2$$

$$y(x^2-2) = 2$$

$$y = \frac{2}{x^2-2} = \frac{2}{(x+\sqrt{2})(x-\sqrt{2})}$$

$$x^2y - 2y = -2$$

$$y = \frac{-2}{x^2-2}$$

$(3,2)$ is on $x^2y - 2y = 12$

$$y = \frac{12}{x^2-2}$$

$(2xy,\ x^2-2)$

(x,y)	$\nabla f(x,y)$
$(0,0)$	$(0,-2)$
$(1,0)$	$(0,-1)$
$(2,0)$	$(0,2)$
$(-1,0)$	$(0,-1)$
$(-2,0)$	$(0,2)$
$(0,1)$	$(0,-2)$
$(1,1)$	$(2,-1)$
$(2,1)$	$(4,2)$
$(-1,1)$	$(-2,-1)$
$(-2,1)$	$(-4,-1)$
$(0,-1)$	$(0,-2)$
$(1,-1)$	$(-2,-1)$
$(2,-1)$	$(-4,2)$
$(-1,-1)$	$(2,-1)$
$(-2,-1)$	$(-4,2)$